VDM recursion in Isabelle/HOL

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Model based design with VDM

- Modelling aspects
  - Abstract data types with invariant and ordering,
  - Function definitions with specification (pre/post/measures),
  - State rich definitions with statements, variable frames and operations.
- Symbolic execution
  - Interpretation of mathematical properties of systems (e.g. possibly with OO and RT features)
  - Limited type scope (e.g. mostly without explicit type bindings; no “x : nat”)
- Testing:
  - Combinatorial traces
  - Code coverage
- Verification:
  - PO generation
  - Quick check (PO disproof) within given domains
  - Various proof support attempts since Jones’ Mural theorem prover halted
  - No symbolic counter example (no model-based / unbounded counter examples)

VDM proof support history since Mural (1991)

1. Maharaj & Bicarregui (shallow) embedding of VDM in PVS (1998)
   - https://www.researchgate.net/publication/2510095_On_the_Verification_of_VDM_Specification_and_Refinement_with_PVS
2. Karabotsos (deep) embedding of VDM (LPF) in Isabelle/HOL (2005)
   - https://spectrum.library.concordia.ca/id/eprint/8505/1/MR10289.pdf
3. Freitas & Woodcock soundness argument/result for VDM proofs in other logics (2008)
   - Unifying Theories of Undefinedness [https://doi.org/10.3233/978-1-58603-976-9-311]
   - Linking Z and VDM: (Z semi-classical) logic prover for VDM (LPF) theorems [https://ieeexplore.ieee.org/document/4492887]
4. Vermolen, Hooman & Larsen (shallow) embedding of VDM in HOL (2010)
   - Hand-crafted proof tactics [https://dl.acm.org/doi/10.1145/1774088.1774608]
5. Freitas & Whiteside (shallow) embedding of VDM in Isabelle/HOL (2014)
   - VDM theorems proved in Z and Isabelle/HOL [https://link.springer.com/chapter/10.1007/978-3-319-06410-9_20]
6. Freitas automated (shallow) embedding of VDM in Isabelle/HOL with proof crafting support (2021)
   - VDM toolkit project [https://github.com/leouk/VDM_Toolkit]

VDM to Isabelle translation strategy 101

- Translation strategy started (2010) as part of the AI4FM project (2010-2014)
  - Attempt to identify proof strategy reuse across provers (Isabelle, ZEves) and models (VDM and Z)
  - Technical report on a translation strategy for most of VDM as shallow embedding
  - Undergraduate course on translating VDM to Isabelle (manually) (2012-2022)
- VDM LPF
  - Pragmatic approach similar to VDMJ's: a L-R logic (e.g. subset of LPF) with "possible" typing rules
- VDM data types
  - Sets, sequences, maps and records map almost directly to Isabelle libraries
  - Numeric types cannot be directly translated given VDM widening-type rules (e.g. "`0 - x`" becomes `int` for `x:nat`)
  - Set and sequence comprehensions are easy; **map comprehension is fiendish**
- VDM functions
  - Direct translation as Isabelle definitions for non-recursive functions
  - Recursive functions support quite limited at first (e.g. only for VDM sequences)

VDM Toolkit Project

https://github.com/leouk/VDM_Toolkit

- Initiative to coalesce various VDM-related developments (e.g. libraries, experiments, etc.) since 2010
- VDM to Isabelle translator (vdm2isa plugin)
  - VDMToolkit : Isabelle proof engineering and support for VDM translation and proof automation
  - exu : VDM style checker and specification reordering (see next talk)
  - vdm2isa : Syntax-driven VDM to Isabelle translator
  - isapog : VDM PO translator and proof script / strategy predictor
  - vdm2isa-lsp : VSCode LSP (editor) + DAP (debugger) integration of plugins
- VDM ANTLR
  - VDM syntax formal definition for parsing, printing, translations, etc.
- VDM Libraries
  - CSV, ISQ, Order, Z-Relations, Dense ranges, Logging, Binary and Matrix arithmetic, General support, etc.
- VDM Annotations
  - Specification profiling; user defined theorems; user defined proof attributes, hints and witnesses.

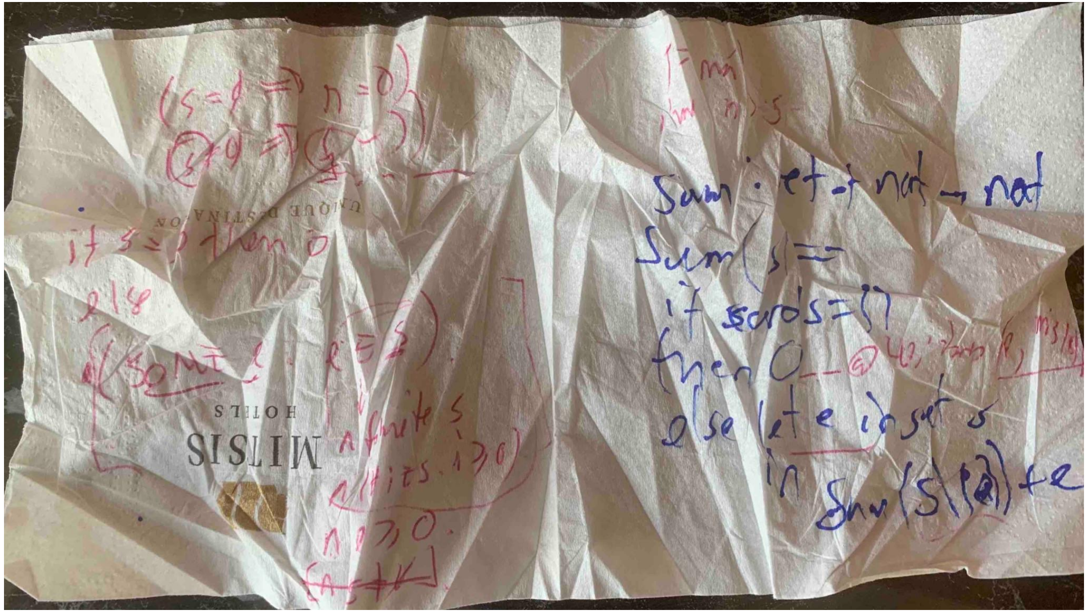
Isabelle recursion principles

- Primitive recursion
  - Reduction rules per type constructors; not possible to have pattern matching
- Total function with automatic proof
  - Implemented on top of primitive recursion with extended pattern matching and non-constructive types
  - Proof obligations:
    - Pattern compatibility (i.e. is the pattern given matchable to input type?);
    - Pattern completeness (i.e. are patterns given exhaustive?);
    - Recursion termination (i.e. are recursive calls well-founded?);
- Total (partial) function with user defined proof
  - Copes with any type and extended pattern as well as partial (non-terminating) functions
  - Partial functions require abstract domain predicates assumptions (psimps-rules) everywhere!
  - Pattern compatibility and completeness proofs are mandatory and with reasonable automation support
  - Termination proofs rely on knowledge of the "*Size Change Termination*" *(SCT)* principle(s)!

Extending translation of VDM recursion to Isabelle

- Translation of recursive functions restricted over VDM sequence parameters only
- Recursive functions over nat parameters are (surprisingly) non-trivial!
- Users requested support for sets, maps, and of course, `nat`!

AKA: PGL's Napkin @ ISoLA22 :-)



VDM-recursion Isabelle-translation caveats

- Isabelle (ℕ, ℤ, ℚ, ℝ) types are defined constructively through different embeddings
  - ℕ: defined inductively over two constructors (e.g. `zero` and `suc n`)
  - ℤ: defined algebraically as a quotient type between two ℕ (e.g. positive and negative parts)
  - ℚ: defined algebraically as a quotient type between two ℤ (e.g. numerator and denominator parts)
  - ℝ: defined algebraically as a "vanishing" Cauchy sequence quotient type
- VDM type widening rules forces the use of maximal type for translation
  - In VDM, for a `x:nat`, "`0-x`" becomes "-x" of type `int`. In Isabelle this is "`0:ℕ`"!
  - Translation encode VDM `nat` as `VDMNat` (ℤ) and ℚ as ℝ
  - Isabelle recursion must be constructive (i.e. will require various transformations for non ℕ)
- VDM recursion over sets and maps are finite; Isabelle sets are infinite and axiomatic
  - Isabelle requires a constructive well-formed recursive relation; some of which are inferrable
  - Isabelle will impose well-formed proof obligations for sets and map domain's finiteness
- VDM measures are not expressive enough for certain recursion patterns (e.g. ack, recursive types, etc.)
  - Some (complex) recursive measures *must* be relational

VDM ℕ-factorial example caveats

Recursive VDM factorial

- Trivial recursive measure

```
factorial: nat -> nat
factorial(n) == if n = 0 then 1 else n * factorial(n - 1) measure n;
```

Isabelle ℕ factorial

- Automatically discovered measure

```
fun fact' :: ‹ℕ ⇒ ℕ› where ‹fact' n = (if n = 0 then 1 else n * (fact' (n - 1)))›
Found termination order: "size <*mlex*> {}"
```

Isabelle VDMNat (ℤ) factorial

- Requires user to prove termination

```
fun fact :: ‹VDMNat ⇒ VDMNat› where "fact n = (if n = 0 then 1 else n * (fact (n - 1)))"
Unfinished subgoals:
(a, 1, <):
 1. ⋀n. n ≠ 0 ⟹ ¦n - 1¦ < ¦n¦
(a, 1, <=):
 1. ⋀n. n ≠ 0 ⟹ nat ¦n - 1¦ ≤ nat ¦n¦
```

Translation recipe for basic types

- Translate (pre/post) specifications
  - Follows usual VDM translation strategy yet creating definition sets
- Translate recursive definition itself
  - Flag controls whether to try Isabelle discovered proofs (e.g. "fun") or user defined (e.g. "functions")
- Infers recursive relation from VDM AST
  - Flag controls whether to generate lemma about well-formedness of inferred recursive relation
  - Presumes inferred relation is within largest well-formed relation from lower bound (e.g. for ℕ bound is 0)

```
definition largest_wf_int_rel :: "ℤ ⇒ (ℤ × ℤ) set" where
"largest_wf_int_rel d = {(z', z). d ≤ z' ∧ z' < z}"
```

- Sets up pattern consistency, completeness and termination proofs
  - Pattern proofs are almost always found by sledgehammer (unless wicked patterns or mutually recursive calls)
  - Termination proof presumes inferred relation is within largest well-formed relation; up to the user otherwise
  - Harder recursive patterns will require users to define recursive relation as VDM annotations (@IsaMeasure, @Witness)

Isabelle Demo (or see paper theory sources)

https://github.com/leouk/VDM_Toolkit/blob/development/plugins/vdm2isa/pub/recursion/RecursiveVDM.thy

https://github.com/leouk/VDM_Toolkit/blob/development/plugins/vdm2isa/pub/recursion/RecursiveVDM.vdmsl

1. Recursion for constructively defined basic types ($\mathbb{N}$)
2. Recursion for non-constructively defined basic types ($\mathbb{Z}$)
3. Recursion for constructively defined structured types (seq) [paper/offline?]
4. Recursion for non-constructively defined structured types (set and maps) [paper/offline]
5. Complex recursion patterns (e.g. Ackerman, Permutation, Takeuchi, etc.) [paper/offline]
6. Simple mutual recursion (e.g. odd and even) [paper]
7. Complex mutual recursion (e.g. N-Queens, Sudoku solvers) [offline]

Discussion

- Results
    - Semi-automated (with proof support) translation recipes/strategies for:
        - VDM some numeric types (nat, nat1, int)
        - VDM structured types (sets, sequences, maps)
    - Automation caters for most common VDM recursive situations
        - Decreasing nat/int, set (or map domain), sequence
- Limitations
    - Recursive VDM types (e.g. VDM records for say linked lists)
    - General upper bound for inferred recursive relations works for `nat`, `int` and (simple) `sets` only.
- Complexities (e.g. will require user-defined auxiliary lemmas and their proofs)
    - Recursion over user-defined types
    - Recursion where recursive relation is outside predefined space
    - Mutual recursion will necessitate handling Isabelle union types

Discussion

- Wish list
  - completeness of VDM patterns to enable handling FMI models
  - including Isabelle/LSP back end IDE run in background to attempt proofs automatically
  - implement complex (and mutual) recursion templates (i.e. POC for now)
  - VSCode code lenses integration (e.g. akin to jUnit testing)
  - ????
  - ????